GEGENBAUER TAU METHODS WITH AND WITHOUT SPURIOUS EIGENVALUES

MARIOS CHARALAMBIDES* AND FABIAN WALEFFE†

Abstract. It is proven that a class of Gegenbauer tau approximations to a 4th order differential eigenvalue problem of hydrodynamic type provide real, negative, and distinct eigenvalues, as is the case for the exact solutions. This class of Gegenbauer tau methods includes Chebyshev and Legendre Galerkin and 'inviscid' Galerkin but does not include Chebyshev and Legendre tau. Rigorous and numerical results show that the results are sharp: positive or complex eigenvalues arise outside of this class. The widely used modified tau approach is proved to be equivalent to the Galerkin method.

Key words. Spurious eigenvalues, Gegenbauer, spectrum, stable polynomials, positive pairs

AMS subject classifications. 65D30, 65L10, 65L15, 65M70, 65N35, 26C10

1. Introduction. The Chebyshev tau method used by Orszag [17] to obtain exponentially accurate solutions of the Orr-Sommerfeld equation yields two eigenvalues with large positive real parts. Such eigenvalues also occur for Stokes modes in a channel given by the fourth order differential equation

$$(D^2-\alpha^2)^2\,u=\lambda(D^2-\alpha^2)u \tag{1.1}$$

with the boundary conditions $u = Du = 0$ at $x = \pm 1$, where λ is the eigenvalue, $u = u(x)$ is the eigenfunction, $D = d/dx$ and α is a real wavenumber. The Stokes eigenvalues λ are real and negative as can be checked by multiplying (1.1) by u^*, the complex conjugate of $u(x)$, and integrating by parts twice using the no-slip boundary conditions. In fact the Stokes spectrum has been known analytically since Rayleigh [7, §26.1]. Yet, the Chebyshev tau method applied to (1.1) yields 2 eigenvalues with large positive real parts, for any order of approximation and for any numerical accuracy. Such eigenvalues are obviously *spurious* for (1.1). Gottlieb and Orszag [11, Chap. 13] introduced the eigenvalue problem

$$\begin{aligned} D^4u&=\lambda\,D^2u \quad \text{in } -1\le x\le 1,\\ u=Du&=0 \quad \text{at} \quad x=\pm 1, \end{aligned} \tag{1.2}$$

as an even simpler 1D model of incompressible fluid flow. This is the $\alpha \to 0$ limit of the eigenvalue problem (1.1) and of the Orr-Sommerfeld equation [17]. For any fixed α, problem (1.2) is also the asymptotic equation for large λ solutions of the Stokes and Orr-Sommerfeld equations. The eigensolutions of (1.2) are known analytically. They consist of even modes $u(x) = 1 - \cos(n\pi x)/\cos(n\pi)$ with $\lambda = -n^2\pi^2$ and odd modes $u(x) = x - \sin(q_n x)/\sin(q_n)$, with $\lambda = -q_n^2$, where $q_n = \tan q_n$, so that $n\pi < q_n < (2n + 1)\pi/2$, $\forall$ integer $n > 0$. The key properties of these solutions are that the eigenvalues are *real, negative, and distinct,* and the even and odd mode eigenvalues interlace. These properties also hold for Stokes modes, the solutions of (1.1), but not for Orr-Sommerfeld modes.

The Chebyshev tau method provides spectrally accurate approximations to the lower magnitude eigenvalues but it also yields two large positive eigenvalues for problems (1.1) and (1.2) [11, Table 13.1]. Those positive eigenvalues are clearly *spurious*

*Department of Business Administration, Frederick University Cyprus, 7 Yianni Frederickou Street, Pallouriotissa, PO Box 24729, 1303 Nicosia, Cyprus (bus.chm@fit.ac.cy).

†Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA (waleffe@math.wisc.edu). This work was supported in part by NSF grant DMS-0204636.

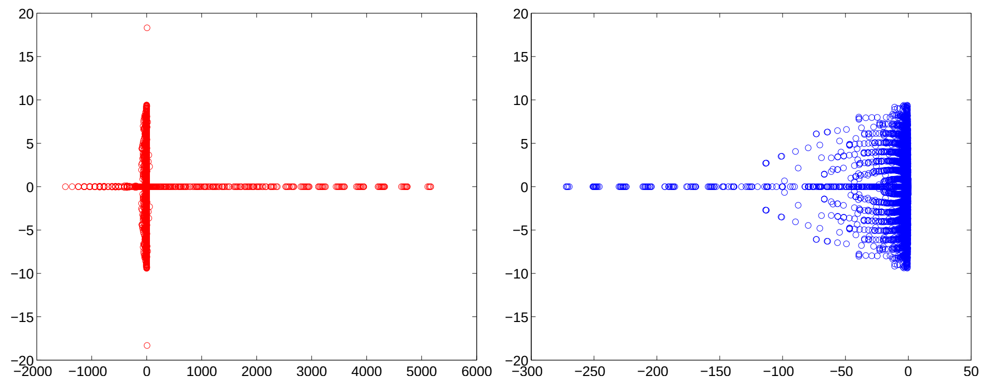

Fig. 1.1. *Eigenvalues of a 3D steady state solution of the Navier-Stokes equations for plane Couette flow [21] computed with Chebyshev tau (left) and Chebyshev Galerkin (right) for identical resolutions (8773 modes after symmetry reductions. The solutions themselves are indistinguishable). Note the difference in horizontal scales. Chebyshev tau produces 274 eigenvalues with positive real parts, 273 of which are spurious. Chebyshev Galerkin returns only one positive eigenvalue, the physical one, equal to 0.03681 at Reynolds number 1000 [21, Fig. 4].*

since it is known that (1.1) and (1.2) should only have negative eigenvalues. The Chebyshev tau method yields 2 spurious eigenvalues for no-slip (*a.k.a.* 'clamped') boundary conditions $u = Du = 0$ at $x = \pm 1$, but none for the free-slip boundary conditions $u = D^2u = 0$ at $x = \pm 1$. The latter problem reduces to the 2nd order problem $D^2v = \lambda v$ with $v(\pm 1) = 0$ for which a class of Jacobi and Gegenbauer tau methods has been proven to yield real, negative and distinct eigenvalues [5, 4]. For mixed boundary conditions, *e.g.* $u(\pm 1) = Du(-1) = D^2u(1) = 0$ there is one spurious eigenvalue (this is a numerical observation).

For a 1D problem such as (1.1) or (1.2), the spurious eigenvalues are easy to recognize and they appear as minor nuisances. Boyd [2, §7.6] even questions the value of distinguishing between 'spurious' and numerically inaccurate eigenvalues. However, in many applications, large *negative* eigenvalues are inconsequential, while 'spurious' *positive* eigenvalues are very significant, and in higher dimensions, spurious eigenvalues are not as easy to pick out and set aside. In a recent application, 3D *unstable* traveling wave solutions of the Navier-Stokes equations were calculated with both free-slip and no-slip boundary conditions, and anything in between, by Newton's method [19, 20]. In that application, the Chebyshev tau method provides hundreds of spurious unstable eigenvalues, depending on resolution and the exact type of boundary conditions, not all of which have very large magnitudes (fig. 1.1, left). A simple change in the *test* functions from Chebyshev polynomials $T_n(x)$ to $(1 - x^2)T_n(x)$ or $(1 - x^2)^2T_n(x)$ eliminates all those spurious eigenvalues (fig. 1.1, right). This, and more, is proven below for the test problem (1.2) in the broader context of Gegenbauer tau methods which include Chebyshev and Legendre tau, and Chebyshev and Legendre Galerkin methods.

Another practical consequence of the spurious eigenvalues is that the Chebyshev tau method is unconditionally unstable when applied to the time-dependent version of (1.1) or (1.2), with $\partial/\partial t$ in place of λ. Such time-dependent problems appear as building blocks in Navier-Stokes simulations of channel-type flows. Gottlieb and Orszag [11, p. 145] proposed a *modified tau* method for the time-dependent problems and proved that the modified method was stable for even solutions. The modified

tau method (sect. 6) is a key idea behind several successful time integration schemes for the Navier-Stokes equations [3, §7.3], [14]. The modified tau method amounts to using 2 more expansion polynomials for the 4th order differential operator on the left hand sides of (1.2) and (1.1) than for the 2nd order operator on the right hand sides. That modified tau method was adapted to eigenvalue problems by Gardner, Trogdon and Douglass [8] and McFadden, Murray and Boisvert [16]. McFadden *et al.* showed the equivalence between the modified Chebyshev tau method and a Chebyshev Galerkin method by direct calculation. Zebib [22] had given numerical evidence that the Galerkin method removed spurious eigenvalues. The modified tau method idea was adapted to the collocation formulation by Huang and Sloan [13].

A heuristic 'explanation' for spurious eigenvalues is that there is a 'mismatch' between the number of boundary conditions applied to the 4th order operator on the left hand side of (1.2) and those applied to the 2nd order operator on the right hand side. That interpretation fits with the modified tau method which uses two more polynomials for the 4th order operator than for the 2nd order operator. However it is incorrect since, while the tau method for Chebyshev polynomials of the 1st kind $T_n(x)$ gives spurious eigenvalues, for instance, the tau method for Chebyshev polynomials of the 2nd kind $U_n(x)$ does not.

All of these various methods are best seen in the context of the Gegenbauer class with residuals weighted by $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$ (sect. 2), where $\gamma = 0$ corresponds to Chebyshev and $\gamma = 1/2$ to Legendre polynomials. Dawkins *et al.*[6] proved existence of spurious positive eigenvalues for (1.2) when $\gamma < 1/2$. The proof is straightforward. For (1.2), the polynomial equation for $\mu = 1/\lambda$ can be derived explicitly (sect. 4). All coefficients of that polynomial are real and positive, except the constant term which is negative when $\gamma < 1/2$. Hence there is one real positive μ, and a 'spurious' positive eigenvalue, when $\gamma < 1/2$ (details are given in sect. 5.2). For $\gamma = 1/2$, the Legendre tau case, the constant term is zero, hence there is one $\mu = 0$ eigenvalue, or a $\lambda = 1/\mu = \infty$ eigenvalue. Perturbation analysis shows that the $\lambda = \infty$ eigenvalues become very large positive eigenvalues for $\gamma < 1/2$ and very large negative eigenvalues for $\gamma > 1/2$. We provide a quicker derivation of those results in section 3. Dawkins *et al.*'s results do not prove that there are no spurious eigenvalues for $\gamma > 1/2$ since there could be complex eigenvalues with positive real parts. In section 5, we prove that the Gegenbauer tau method applied to (1.2) provides eigenvalues that are *real, negative, and distinct* when $1/2 < \gamma \le 7/2$. This provides a complete characterization of the Gegenbauer tau spectrum for problem (1.2). Numerical calculations confirm that the range $1/2 < \gamma \le 7/2$ is sharp. Spurious positive eigenvalues exist for $\gamma < 1/2$ [6] and complex eigenvalues arise for $\gamma > 7/2$, for sufficiently high polynomial order. In section 6, we prove that the modified tau method is mathematically equivalent to the Galerkin approach.

Obviously, $\gamma = 1/2$ is a critical value for the weight function $(1-x^2)^{\gamma-1/2}$. The boundaries $x = \pm 1$ have infinite weight for $\gamma < 1/2$, and zero weight for $\gamma > 1/2$, but we do not know a valid heuristic explanation for 'spurious' eigenvalues beyond that observation, if one exists. Section 3 provides further insights into the nature of the spurious eigenvalues and gives *some* support for the view that 'spurious' and numerically inaccurate eigenvalues are related. In figure 1.1, *for fixed resolution*, the 273 spurious eigenvalues for $\gamma = 0$ (Chebyshev tau), escape to $+\infty$ as $\gamma \nearrow 1/2$ (Legendre tau). They come back from $-\infty$ as γ increases beyond $1/2$. Thus, there is indeed a connection between large positive eigenvalues and large negative eigenvalues, but whether we have 'spurious' positive eigenvalues or inconsequential very negative

eigenvalues is sharply controlled by γ, irrespective of the order of approximation n (see also eqn. (3.6)).

2. Tau and Galerkin methods. DEFINITION 2.1. *A Gegenbauer tau method approximates the solution $u(x)$ of a differential equation in $-1 \leq x \leq 1$ by a polynomial of degree n, $u_n(x)$, that satisfies the m boundary conditions exactly. The remaining $n+1-m$ polynomial coefficients are determined by imposing that the residual be orthogonal to all polynomials of degree $n-m$ (or less) with respect to the Gegenbauer weight $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$, with $\gamma > -1/2$.*

For problem (1.2), the residual

$$R_{n-2}(x) \equiv \lambda D^2 u_n(x) - D^4 u_n(x) \tag{2.1}$$

is a polynomial of degree $n-2$ in x. The polynomial approximation $u_n(x)$ is determined from the 4 boundary conditions $u_n(\pm 1) = Du_n(\pm 1) = 0$ and the requirement that $R_{n-2}(x)$ is orthogonal to *all* polynomials $q_{n-4}(x)$ of degree $n-4$ or less with respect to the weight function $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2} \geq 0$ in the interval $(-1, 1)$

$$\int_{-1}^{1} R_{n-2}(x)\, q_{n-4}(x)\, W^{(\gamma)}(x) dx = 0, \quad \forall\, q_{n-4}(x). \tag{2.2}$$

This provides $n-3$ equations which together with the 4 boundary conditions yield $n+1$ equations for the $n+1$ undetermined coefficients in the polynomial approximation $u_n(x)$. For the Gegenbauer weight function $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$, the residual can be written explicitly as

$$R_{n-2}(x) = \tau_0 \lambda\, G_{n-2}^{(\gamma)}(x) + \tau_1 \lambda\, G_{n-3}^{(\gamma)}(x) \tag{2.3}$$

for some x-independent coefficients τ_0 and τ_1, where $G_n^{(\gamma)}(x)$ is the Gegenbauer polynomial of degree n. This follows from orthogonality of the Gegenbauer polynomials in $-1 < x < 1$ with respect to the weight $(1-x^2)^{\gamma-1/2}$ which implies orthogonality of the Gegenbauer polynomial of degree k to *any* polynomial of degree $k-1$ or less with respect to that weight function.

Gegenbauer (*a.k.a.* ultraspherical) polynomials are a special subclass of the Jacobi polynomials [1]. The latter are the most general class of polynomial solutions of a Sturm-Liouville eigenproblem that is singular at ± 1, as required for faster than algebraic convergence [3]. Gegenbauer polynomials are the most general class of polynomials with the odd-even symmetry $G_n^{(\gamma)}(x) = (-1)^n G_n^{(\gamma)}(-x)$. This is a one parameter family of polynomials, with the parameter $\gamma > -1/2$. Chebyshev polynomials correspond to $\gamma = 0$ and Legendre polynomials to $\gamma = 1/2$. We use a (slightly) non-standard normalization of Gegenbauer polynomials since the standard normalization [1] is singular in the Chebyshev case. Some key properties of Gegenbauer polynomials used in this paper are given in Appendix A. Note that if $\lambda = 0$ then, from (2.1), the residual $R_{n-2}(x)$ must be a polynomial of degree $n-4$ implying that $\tau_0 = \tau_1 = 0$ in (2.3) and $D^4 u_n(x) = 0$ for all x in $(-1, 1)$. The boundary conditions $u_n(\pm 1) = Du_n(\pm 1) = 0$ then imply that $u_n(x) = 0$ for all x in $[-1, 1]$, the trivial solution. Hence we can assume that $\lambda \neq 0$ in the Gegenbauer tau method applied to (1.2).

Following common usage [11, 3, 22, 16], we have

DEFINITION 2.2. *A Gegenbauer Galerkin method approximates the solution $u(x)$ of a differential equation in $-1 \leq x \leq 1$ by a polynomial of degree n, $u_n(x)$, that*

satisfies the m boundary conditions exactly. The remaining $n+1-m$ polynomial coefficients are determined by imposing that the residual be orthogonal to all polynomials of degree n (or less) that satisfy the homogeneous boundary conditions, with respect to the Gegenbauer weight $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$, with $\gamma > -1/2$.

Strictly speaking, this a *Petrov-Galerkin* method since the test functions are not identical to the trial functions because of the Gegenbauer weight $(1-x^2)^{\gamma-1/2}$ [3].

For problem (1.2), $u_n(x)$ is determined from the boundary conditions $u_n(\pm 1) = Du_n(\pm 1) = 0$ and orthogonality, with respect to weight $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$, of the residual (2.1) to all polynomials of degree n that vanish together with their derivative at $x = \pm 1$. Such polynomials can be written as $(1-x^2)^2 q_{n-4}(x)$ where $q_{n-4}(x)$ is an arbitrary polynomial of degree $n-4$, and the weighted residual equations read

$$\int_{-1}^{1} R_{n-2}(x)\,(1-x^2)^2 q_{n-4}(x)\,W^{(\gamma)}(x)dx = 0, \quad \forall\, q_{n-4}(x). \tag{2.4}$$

The Gegenbauer Galerkin method is therefore equivalent to the tau method for the weight $W^{(\gamma+2)}(x) = (1-x^2)^2 W^{(\gamma)}(x)$ and its residual has the explicit form

$$R_{n-2}(x) = \tau_0 \lambda\, G_{n-2}^{(\gamma+2)}(x) + \tau_1 \lambda\, G_{n-3}^{(\gamma+2)}(x). \tag{2.5}$$

So a *Chebyshev (or Legendre) Galerkin* method for clamped boundary conditions, $u_n(\pm 1) = Du_n(\pm 1) = 0$, is in fact a tau method for Chebyshev (or Legendre) polynomials of the *3rd kind* (proportional to the 2nd derivative of Chebyshev (or Legendre) polynomials (A.5)). Since we consider a range of the Gegenbauer parameter γ, the Gegenbauer tau method also includes some Gegenbauer Galerkin methods.

This suggests an intermediate method where the test functions are polynomials that vanish at $x = \pm 1$ (inviscid boundary conditions only).

Definition 2.3. *The Gegenbauer 'inviscid Galerkin' method determines $u_n(x)$ from the 4 boundary conditions $u_n(\pm 1) = Du_n(\pm 1) = 0$ and orthogonality of the residual to all polynomials of degree $n-2$ that vanish at $x = \pm 1$, with respect to the weight function $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$.*

Such test polynomials can be written in the form $(1-x^2)q_{n-4}(x)$ where $q_{n-4}(x)$ is an arbitrary polynomial of degree $n-4$, so the weighted residual equations read

$$\int_{-1}^{1} R_{n-2}(x)\,(1-x^2) q_{n-4}(x)\,W^{(\gamma)}(x)dx = 0, \quad \forall\, q_{n-4}(x). \tag{2.6}$$

The *Gegenbauer inviscid Galerkin* method is therefore equivalent to a Gegenbauer tau method with weight $W^{(\gamma+1)}(x)$ and its residual for (1.2) is

$$R_{n-2}(x) = \tau_0 \lambda\, G_{n-2}^{(\gamma+1)}(x) + \tau_1 \lambda\, G_{n-3}^{(\gamma+1)}(x). \tag{2.7}$$

Thus, a Chebyshev (or Legendre) inviscid Galerkin method is a tau method for Chebyshev (or Legendre) polynomials of the *2nd kind*. Since we consider a range of the Gegenbauer parameter γ, the Gegenbauer tau method also includes some Gegenbauer Inviscid Galerkin methods.

For completeness, we list the *collocation* approach, where $u_n(x)$ is determined from the boundary conditions $u_n(\pm 1) = Du_n(\pm 1) = 0$ and enforcing $R_{n-2}(x_j) = 0$ at the $n-3$ interior Gauss-Lobatto points x_j such that $DG_{n-2}(x_j) = 0$, $j = 1, ..., n-3$, [3, §2.2]. The residual (2.1) has the form [10, eqn. (4.5)]

$$R_{n-2}(x) = (A + Bx)\, DG_{n-2}(x), \tag{2.8}$$

for some A and B independent of x. That residual can be written in several equivalent forms by using the properties of Gegenbauer polynomials (appendix A). We do not have rigorous results for the collocation method.

3. Legendre and near-Legendre tau cases. Here we provide a quicker and more complete derivation of earlier results [6] about spurious eigenvalues for the Legendre and near-Legendre tau case. This section provides a useful technical introduction to the problem but is not necessary to derive the main results of this paper. Dawkins *et al.*[6], focusing only on even modes, use the monomial basis x^{2k} to derive an explicit form for the generalized eigenvalue problem $Aa = \lambda Ba$ for the Legendre tau method. In the monomial basis, the matrix A is upper triangular and nonsingular, and the matrix B is upper Hessenberg but its first row is identically zero, hence there exists one infinite eigenvalue. A perturbation analysis is used to show that the infinite eigenvalue of the Legendre tau method becomes a large positive eigenvalue for Gegenbauer tau methods with $\gamma < 1/2$ and a large negative eigenvalue for $\gamma > 1/2$.

In the Legendre tau method, the polynomial approximation $u_n(x)$ of degree n to problem (1.2) satisfies the 4 boundary conditions $u_n = Du_n = 0$ at $x = \pm 1$. Thus $u_n(x) = (1-x^2)^2 p_{n-4}(x)$ and the polynomial $p_{n-4}(x)$ is determined from the weighted residual equations (2.2) with $\gamma = 1/2$ and $W^{(1/2)}(x) = 1$,

$$\int_{-1}^{1} \left(\mu D^4 u_n - D^2 u_n\right)\, q_{n-4}(x) dx = 0, \quad \forall\, q_{n-4}(x). \tag{3.1}$$

The mathematical problem is fully specified, except for an arbitrary multiplicative constant for $u_n(x)$. Choosing various polynomial bases for $p_{n-4}(x)$ and $q_{n-4}(x)$ will lead to distinct matrix problems but those problems are all similar to each other and provide exactly the same eigenvalues, in exact arithmetic.

We use the bases $G_l^{(5/2)}(x)$ for $p_{n-4}(x)$ and $G_k^{(1/2)}(x)$ for $q_{n-4}(x)$, with $k, l = 0, \ldots, n-4$, where $G_n^{(\gamma)}(x)$ is the Gegenbauer polynomial of degree n for index γ (see appendix A and recall that $G_k^{(1/2)}(x) = P_k(x)$ are Legendre polynomials). Thus we write $u_n(x) = \sum_{l=0}^{n-4} a_l\ (1-x^2)^2 G_l^{(5/2)}(x)$, for some $n-3$ coefficients a_l to be determined. The tau equations (3.1) provide the matrix eigenproblem $\mu Aa = Ba$, or $\mu \sum_{l=0}^{n-4} A(k,l) a_l = \sum_{l=0}^{n-4} B(k,l) a_l$ with

$$A(k,l) = \int_{-1}^{1} D^4 \left[(1-x^2)^2 G_l^{(5/2)}(x)\right]\ G_k^{(1/2)}(x)\, dx, \tag{3.2}$$

$$B(k,l) = \int_{-1}^{1} D^2 \left[(1-x^2)^2 G_l^{(5/2)}(x)\right]\ G_k^{(1/2)}(x)\, dx, \tag{3.3}$$

for $k, l = 0, \ldots, n-4$. Using (B.1), these expressions simplify to

$$A(k,l) = \mathcal{C}_l \int_{-1}^{1} \left[D^2 G_{l+2}^{(1/2)}(x)\right]\ G_k^{(1/2)}(x)\, dx, \tag{3.4}$$

$$B(k,l) = \mathcal{C}_l \int_{-1}^{1} G_{l+2}^{(1/2)}(x)\ G_k^{(1/2)}(x)\, dx, \tag{3.5}$$

where $\mathcal{C}_l = \frac{1}{15}(l+1)(l+2)(l+3)(l+4)$.

Since the Legendre polynomials $G_n^{(1/2)}(x) = P_n(x)$ are orthogonal with respect to the unit weight, equation (3.5) yields that $B(k,l) \propto \delta_{k,l+2}$, where $\delta_{k,l+2}$ is the

Kronecker delta, so that $B(0,l) = B(1,l) = 0$ for all l and B has non zero elements only on the second subdiagonal. For $A(k,l)$, use (A.10) to express $D^2 G_{l+2}^{(1/2)}(x)$ as a linear combination of $G_l^{(1/2)}(x)$, $G_{l-2}^{(1/2)}(x)$, etc. Orthogonality of the Legendre polynomials $G_n^{(1/2)}(x)$ then implies that $A(k,l)$ is upper triangular with non-zero diagonal elements. Hence A is non-singular while the nullspace of B is two-dimensional. The eigenvalue problem $\mu A a = B a$ therefore has two $\mu = 0$ eigenvalues. Since the only non zero elements of B consists of the sub-diagonal $B(l+2,l)$, the two *right* eigenvectors corresponding to $\mu = 0$ are $a = [0,\ldots,0,1]^T$ and $[0,\ldots,0,1,0]^T$. In other words,

$$u_n(x) = (1-x^2)^2\, G_{n-4}^{(5/2)}(x) \quad \text{and} \quad u_n(x) = (1-x^2)^2\, G_{n-5}^{(5/2)}(x) \tag{3.6}$$

satisfy the boundary conditions and the tau equations (3.1) with $\mu = 0 = 1/\lambda$, for all $n \geq 5$. One mode is even, the other one is odd. Likewise, the *left* eigenvectors $b^T = [1, 0\ldots,0]$ and $[0,1,0\ldots,0]$, satisfy $\mu b^T A = b^T B$ with $\mu = 0$. These results are for the Legendre case and $\mu = 0$ corresponds to $\lambda = 1/\mu = \infty$.

Now consider the Gegenbauer tau equations for $\gamma - 1/2 = \epsilon$ with $|\epsilon| \ll 1$, the near-Legendre case. The equations are (3.1) but with the extra weight factor $W^{(\gamma)}(x) = (1-x^2)^\epsilon$ inside the integral. We can figure out what happens to the $\mu = 0$ eigenvalues of the $\epsilon = 0$ Legendre case by perturbation. The matrices A and B and the left and right eigenvectors, denoted a and b respectively, as well as the eigenvalue μ now depend on ϵ. Let

$$A = A_0 + \epsilon A_1 + O(\epsilon^2), \qquad B = B_0 + \epsilon B_1 + O(\epsilon^2), \tag{3.7}$$

$$a = a_0 + \epsilon a_1 + O(\epsilon^2), \qquad b = b_0 + \epsilon b_1 + O(\epsilon^2), \tag{3.8}$$

$$\mu = \mu_0 + \epsilon \mu_1 + O(\epsilon^2), \tag{3.9}$$

where A_0 and B_0 are the matrices obtained above in (3.4) and (3.5) for $\epsilon = 0$ while a_0 and b_0^H are the corresponding right and left eigenvectors so that $\mu_0 A_0 a_0 = B_0 a_0$ and $\mu_0 b_0^H A_0 = b_0^H B_0$. Substituting these ϵ-expansions in the eigenvalue equation $\mu A a = B a$ and canceling out the zeroth order term, we obtain

$$\mu_1 A_0 a_0 + \mu_0 A_1 a_0 + \mu_0 A_0 a_1 = B_1 a_0 + B_0 a_1 + O(\epsilon). \tag{3.10}$$

Multiplying by b_0^H cancels out the $\mu_0 b_0^H A_0 a_1 = b_0^H B_0 a_1$ terms so we obtain

$$\mu_1 = \frac{b_0^H B_1 a_0 - \mu_0 b_0^H A_1 a_0}{b_0^H A_0 a_0}. \tag{3.11}$$

This expression is general but simplifies further since we are interested in the perturbation of the zero eigenvalues $\mu_0 = 0$. This expression for μ_1 is quite simple since b_0, a_0 and A_0 are the zeroth order objects. All we need to compute when $\mu_0 = 0$ is the first order correction B_1 to the matrix B. But since a_0 and b_0 have only one non-zero component as given at the end of the previous paragraph, we only need to calculate two components of the B matrix. For n even, all that is needed are the first order corrections to $B(0, n-4)$ for the even mode and to $B(1, n-5)$ for the odd mode. For n odd, we need $B(0, n-5)$ for the even mode and $B(1, n-4)$ for the odd mode, however since even and odd modes decouple in this problem, it suffices to compute both even and odd modes in only one case of n even or odd. The matrix elements in the $\epsilon \neq 0$ cases are still given by (3.4) and (3.5) but with the extra $(1-x^2)^\epsilon$ weight

factor inside the integrals. Since $G_1^{(1/2)}(x) = x$ and $G_n^{(1/2)}(x) = P_n(x)$, the Legendre polynomial of degree n, we obtain

$$B(0, n-4) = \mathcal{C}_{n-4} \int_{-1}^{1} P_{n-2}(x)\, (1-x^2)^{\epsilon}\, dx = \epsilon B_1(0, n-4) + O(\epsilon^2), \tag{3.12}$$

$$B(1, n-5) = \mathcal{C}_{n-5} \int_{-1}^{1} x\, P_{n-3}(x)\, (1-x^2)^{\epsilon}\, dx = \epsilon B_1(1, n-5) + O(\epsilon^2), \tag{3.13}$$

with $\mathcal{C}_l$ as defined in (3.5). The integrals are readily evaluated and details are provided in appendix B. Using (B.2), (B.5) and (B.8) we obtain for the even mode (for n even) that

$$\mu_1 = \frac{B_1(0, n-4)}{A_0(0, n-4)} = \frac{-4}{(n-2)^2(n-1)^2}. \tag{3.14}$$

This matches the formula in Dawkins *et al.*[6, page 456] since their $2N = n-4$ and $2\nu - 1 = 2\epsilon$. Likewise, using (B.7), and (B.9) for the odd mode (with n even) yields

$$\mu_1 = \frac{B_1(1, n-5)}{A_0(1, n-5)} = \frac{-4}{(n-4)^2(n-1)^2}. \tag{3.15}$$

Again, if n is odd then μ_1 for the even mode is given by (3.14) but with $n-1$ in lieu of n. Likewise for n odd, the odd mode is given by (3.15) with $n+1$ in lieu of n. Finally, since $\lambda = 1/\mu$, the $\lambda = \infty$ eigenvalues in the Legendre tau case, become $\lambda = 1/(\epsilon\mu_1 + O(\epsilon^2)) \sim 1/(\epsilon\mu_1)$ in the near-Legendre cases. From (3.14) and (3.15), these eigenvalues will be $O(n^4/\epsilon)$. Furthermore they will be positive when $\epsilon < 0$ (*i.e.spurious* when $\gamma < 1/2$) but negative when $\epsilon > 0$.

4. Characteristic Polynomials. For the model problem (1.2), we can bypass the matrix eigenproblem of section 3 to directly derive the characteristic polynomial for the eigenvalues $\mu = 1/\lambda$. To do so, invert equation (2.1) to express the polynomial approximation $D^2 u_n(x)$ in terms of the residual $R_{n-2}(x)$

$$D^2 u_n(x) = \mu \sum_{k=0}^{\infty} \mu^k D^{2k} R_{n-2}(x) \tag{4.1}$$

where $\mu = 1/\lambda$. The inversion (4.1) follows from application of the geometric (Neumann) series for $(1 - \mu D^2)^{-1} = \sum_{k=0}^{\infty} \mu^k D^{2k}$ which terminates since $R_{n-2}(x)$ is a polynomial. Thus, $u_n(x)$ can be computed in terms of the unknown tau coefficients by double integration of (4.1) and application of the boundary conditions. We can assume that $\lambda \neq 0$ because $\lambda = 0$ with $u_n(\pm 1) = Du_n(\pm 1) = 0$ necessarily corresponds to the trivial solution $u_n(x) = 0$, $\forall x$ in $[-1, 1]$, as noted in the previous section.

The Gegenbauer polynomials are even in x for n even and odd for n odd (A.8). The symmetry of the differential equation (1.2) and of the Gegenbauer polynomials allows decoupling of the discrete problem into even and odd solutions. This parity reduction leads to simpler residuals and simpler forms for the corresponding characteristic polynomials. The residual in the parity-separated Gegenbauer case contains only one term

$$R_{n-2}(x) = \tau_0 \lambda\, G_{n-2}^{(\gamma)}(x), \tag{4.2}$$

instead of (2.3), where $G_n^{(\gamma)}(x)$ is the Gegenbauer polynomial of degree n with n even for even solutions and odd for odd solutions. Substituting (4.2) in (4.1) and renormalizing $u_n(x)$ by τ_0 gives

$$D^2 u_n(x) = \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-2}^{(\gamma)}(x). \tag{4.3}$$

For $\gamma > 1/2$, the identity (A.5) in the form $2\gamma G_{n-2}^{(\gamma)}(x) = D G_{n-1}^{(\gamma-1)}(x)$ can be used to write (4.3) in the form

$$D^2 u_n(x) = \frac{1}{2\gamma} \sum_{k=0}^{\infty} \mu^k D^{2k+1} G_{n-1}^{(\gamma-1)}(x), \tag{4.4}$$

which integrates to

$$D u_n(x) = \frac{1}{2\gamma} \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(x) + C, \tag{4.5}$$

where C is an arbitrary constant.

4.1. Even Solutions. For even solutions $u_n(x) = u_n(-x)$, n is even and $Du_n(x)$ is odd so $C = 0$ in (4.5). The boundary condition $Du_n(1) = 0$ gives the characteristic equation for μ (for n even and $\gamma > 1/2$)

$$\sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(1) = 0. \tag{4.6}$$

4.2. Odd Solutions. For odd solutions, $u_n(x) = -u_n(-x)$, n is odd and the boundary condition $Du_n(1) = 0$ requires that

$$C = -\frac{1}{2\gamma} \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(1). \tag{4.7}$$

Substituting this C value in (4.5) and integrating gives

$$2\gamma\, u_n(x) = \sum_{k=0}^{\infty} \mu^k D^{2k-1} G_{n-1}^{(\gamma-1)}(x) - x \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(1) \tag{4.8}$$

where we must define

$$D^{-1} G_{n-1}^{(\gamma-1)}(x) = \int_0^x G_{n-1}^{(\gamma-1)}(s)\, ds = \frac{G_n^{(\gamma-1)}(x) - G_{n-2}^{(\gamma-1)}(x)}{2(n+\gamma-2)} \tag{4.9}$$

since $u_n(x)$ and n are odd, where we have used (A.10) to evaluate the integral and the symmetry (A.8) so that $G_n(0) = G_{n-2}(0) = 0$ for n odd. The boundary condition $u_n(1) = 0$ yields the characteristic polynomial equation (for n odd and $\gamma > 1/2$)

$$\sum_{k=0}^{\infty} \mu^k D^{2k-1} G_{n-1}^{(\gamma-1)}(1) - \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(1) = 0. \tag{4.10}$$

For $\gamma > 3/2$, we can use identity (A.5) in the form $2(\gamma-1)G_{n-1}^{(\gamma-1)}(x) = DG_n^{(\gamma-2)}(x)$, to write the characteristic equation (4.10) as

$$\sum_{k=0}^{\infty} \mu^k D^{2k} G_n^{(\gamma-2)}(1) - \sum_{k=0}^{\infty} \mu^k D^{2k+1} G_n^{(\gamma-2)}(1) = 0. \tag{4.11}$$

For $1/2 < \gamma \leq 3/2$, this cannot be used since $\gamma - 2 < -1/2$, but using (4.9) for the D^{-1} term in the first sum, the characteristic equation (4.10) can be written

$$\mu \sum_{k=0}^{\infty} \mu^k D^{2k+1} G_{n-1}^{(\gamma-1)}(1) - \frac{G_{n-2}^{(\gamma-1)}(1) - G_n^{(\gamma-1)}(1)}{2(n+\gamma-2)} - \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma-1)}(1) = 0. \tag{4.12}$$

5. Zeros of Characteristic Polynomials. Here we prove that the zeros of the characteristic polynomial equations (4.6) and (4.10) are real, negative, and distinct for $1/2 < \gamma \leq 7/2$. Some background material is needed.

5.1. Stable Polynomials and the Hermite Biehler Theorem. A polynomial $p(z)$ is *stable* if and only if all its zeros have negative real parts. Stable polynomials can arise as characteristic polynomials of a numerical method applied to a differential equation as in [5] for $Du = \lambda u$ with $u(1) = 0$ and in other dynamical systems applications. The characterization of stable polynomials that is most useful here is given by [12], [18, p.197],

THEOREM 5.1. *The Hermite-Biehler Theorem. The real polynomial* $p(z) = \Omega(z^2) + z\Theta(z^2)$ *is stable if and only if* $\Omega(\mu)$ *and* $\Theta(\mu)$ *form a positive pair.*

DEFINITION 5.2. *Two real polynomials* $\Omega(\mu)$ *and* $\Theta(\mu)$ *of degree* n *and* $n-1$ *(or* n*) respectively, form a positive pair if:*

(a) the roots $\mu_1, \mu_2, \ldots, \mu_n$ *of* $\Omega(\mu)$ *and* $\mu_1', \mu_2', \ldots, \mu_{n-1}'$ *(or* $\mu_1', \mu_2', \ldots, \mu_n'$*) of* $\Theta(\mu)$ *are all distinct, real and negative.*

(b) the roots interlace as follows: $\mu_1 < \mu_1' < \mu_2 < \cdots < \mu_{n-1}' < \mu_n < 0$ *(or* $\mu_1' < \mu_1 < \cdots < \mu_n' < \mu_n < 0$ *)*

(c) the highest coefficients of $\Omega(\mu)$ *and* $\Theta(\mu)$ *are of like sign.*

We will use the following theorem about positive pairs [15, p198],

THEOREM 5.3. *Any nontrivial real linear combination of two polynomials of degree* n *(or* n *and* $n-1$*) with interlacing roots has real roots.* (Since such a linear combination changes sign $n-1$ times along the real axis, it has $n-1$ real roots. Since it is a real polynomial of degree n, the remaining root is real also.)

5.2. Eigenvalues for even modes. In [5] and [4] we study the Gegenbauer tau method for $D^2u = \lambda u$ with $u(\pm 1) = 0$, which leads to the characteristic polynomials $\sum_{k=0}^{\infty} \mu^k D^{2k} G_n^{(\gamma)}(1)$. The derivation of that result is entirely similar to that in sections 2 and 4. The strategy to prove that the Gegenbauer tau method for that 2nd order problem has real, negative and distinct roots is to show stability of the polynomial

$$p(z) = \sum_{k=0}^{n} z^k \, D^k G_n^{(\gamma)}(1) \tag{5.1}$$

for $-1/2 < \gamma \leq 3/2$ then to use the Hermite Biehler Theorem to deduce that

THEOREM 5.4. *For* $-1/2 < \gamma \leq 3/2$*, the polynomials*

$$\Omega_n^{(\gamma)}(\mu) = \sum_{k=0}^{\infty} \mu^k D^{2k} G_n^{(\gamma)}(1) \qquad \textit{and} \qquad \Theta_n^{(\gamma)}(\mu) = \sum_{k=0}^{\infty} \mu^k D^{2k+1} G_n^{(\gamma)}(1) \tag{5.2}$$

form a positive pair. From (A.5) this is equivalent to stating that the polynomials

$$\Omega_n^{(\gamma)}(\mu) = \sum_{k=0}^{\infty} \mu^k D^{2k} G_n^{(\gamma)}(1) \qquad \text{and} \qquad \Omega_{n-1}^{(\gamma+1)}(\mu) = \sum_{k=0}^{\infty} \mu^k D^{2k} G_{n-1}^{(\gamma+1)}(1) \tag{5.3}$$

also form a positive pair. Combining the γ and $\gamma+1$ ranges in (5.3) yields that $\Omega_n^{(\gamma)}(\mu)$ has real, negative, and distinct roots for $-1/2 < \gamma \leq 5/2$. Stability of (5.1) is proven in [5, Theorem 1] for the broader class of Jacobi polynomials. The basic ideas of the proof are along the lines of Gottlieb [9] and Gottlieb and Lustman's work [10] on stability of the Chebyshev collocation method for the 1st and 2nd order operator.

For the 4th order problem (1.2), $D^4u = \lambda D^2 u$ with $u(\pm 1) = Du(\pm 1) = 0$, the Gegenbauer tau method gives the characteristic polynomial (4.6) for even solutions. This is the polynomial $\Omega_{n-1}^{(\gamma-1)}(\mu)$ of (5.3) that appears for the 2nd order problem [4, 5] and is known to have real, negative, and distinct eigenvalues for $-1/2 < \gamma - 1 \leq 5/2$, that is for $1/2 < \gamma \leq 7/2$. Hence, it follows directly from (4.6) and theorem 5.4 that the Gegenbauer tau approximation for even solutions of problem (1.2), $D^4u = \lambda D^2u$ with $u(\pm 1) = Du(\pm 1) = 0$, has real, negative, and distinct eigenvalues for $1/2 < \gamma \leq 7/2$.

This result is sharp. For $\gamma > 7/2$ and sufficiently large n, our numerical computations show that the polynomial has a pair of complex eigenvalues. For $\gamma < 1/2$ the polynomial (4.6) has a real positive eigenvalue as first proven in [6]. The proof goes as follows. For $\gamma < 1/2$, we cannot use (4.5) since $\gamma - 1$ is below the range of definition of Gegenbauer polynomials (appendix A). Instead, integrate (4.3) and use identity (A.10) to obtain $\int_0^1 G_{n-2}^{(\gamma)} dx = (G_{n-1}^{(\gamma)}(1) - G_{n-3}^{(\gamma)}(1))/(2(\gamma + n - 2))$ since $G_{n-1}(0) = G_{n-3}(0) = 0$ for n even (A.8). Using formula (A.11), one shows that $G_n^{(\gamma)}(1)$ increases with n if $\gamma > 1/2$ but decreases with n if $\gamma < 1/2$. Hence, the constant term is negative for $\gamma < 1/2$ while all the other coefficients of the characteristic polynomial can be shown to be positive using (A.5) and (A.11). Therefore, there is one real positive eigenvalue as proved in [6]. For $\gamma = 1/2$, the Legendre case, the constant term vanishes and there is a $\mu = 0$ ($\lambda = \infty$) eigenvalue as established in section 3.

REMARK 1. *Exact even solutions of $D^4u = \lambda D^2u$ with $u(\pm 1) = Du(\pm 1) = 0$ obey $D^3u = \lambda Du + C$ with $C = 0$, since D^3u and Du are odd. Thus even solution eigenvalues of (1.2) are equal to the eigenvalues for odd solutions of the 2nd order problem $D^2w = \lambda w$ with $w = 0$ at $x = \pm 1$, with $w = Du$. The Gegenbauer tau version of this property is that eigenvalues for even Gegenbauer tau solutions of (1.2) of order n (even) and index γ are equal to the eigenvalues for odd Gegenbauer tau solutions of $D^2w = \lambda w$, $w(\pm 1) = 0$ of order $n - 1$ (odd) and index $\gamma - 1$. This follows directly from (4.6) and [5, 4].*

5.3. Eigenvalues for odd modes. The reduction of the 4th order problem (1.2) to the 2nd order problem does not hold for odd modes which have the characteristic equation (4.10). In fact, all previous theoretical work focused only on the even modes [6, 11]. The same general strategy as for the even case led us to prove the stability of a shifted version of polynomial (5.1).

THEOREM 5.5. *Let $G_n^{(\gamma)}(x)$ denote the non standard Gegenbauer polynomial of degree n as defined in appendix A, then the polynomial*

$$p_n^{(\gamma)}(z) = \frac{G_{n-1}^{(\gamma)}(1) - G_{n+1}^{(\gamma)}(1)}{2(n+\gamma)} + \sum_{k=0}^{n} z^k\, D^k G_n^{(\gamma)}(1) \tag{5.4}$$

is stable for $-1/2 < \gamma \leq 1/2$.

The proof is elementary but technical, it is given in Appendix C. We also need the following simple lemma. This lemma will help us determine the sign of the coefficients of the characteristic polynomials.

LEMMA 5.6. *With* $G_n^{(\gamma)}(x)$ *as defined in appendix A, the expression*

$$D^{k+1}G_n^{(\gamma)}(1) - D^k G_n^{(\gamma)}(1) \geq 0 \tag{5.5}$$

for $k = 0, \ldots, n-1$ *and* $\gamma > -1/2$.

Proof. From (A.13)

$$D^k G_n^{(\gamma)}(1) = \frac{2^{k-1}\Gamma(\gamma+k)\Gamma(n+2\gamma+k)}{(n-k)!\Gamma(\gamma+1)\Gamma(2\gamma+2k)}, \tag{5.6}$$

where $\Gamma(z)$ is the standard gamma function. For $k > 0$ and given that $\gamma > -1/2$, the sign of the above expression is positive since all individual terms are positive. In the $k = 0$ case, the sign of the expression is determined by the terms $\frac{\Gamma(\gamma)}{\Gamma(2\gamma)}$ since all other terms are positive. If $-1/2 < \gamma < 0$ both numerator and denominator are negative and thus their ratio is positive. If $\gamma > 0$ the two terms are positive and thus again their ratio is positive. For $\gamma = 0$, a simple limiting argument shows positiveness again, in fact from (A.14), $G_n^{(0)}(1) = T_n(1)/n = 1/n$.

Taking the ratio $D^{k+1}G_n^{(\gamma)}(1)/D^k G_n^{(\gamma)}(1)$ and making some simplifications gives

$$\frac{D^{k+1}G_n^{(\gamma)}(1)}{D^k G_n^{(\gamma)}(1)} = \frac{(2\gamma+n+k)(n-k)}{(2\gamma+2k+1)}. \tag{5.7}$$

Since $k \leq n-1$ then $2\gamma + 2k + 1 \leq 2\gamma + (n-1) + k + 1 = 2\gamma + n + k$. Thus

$$\frac{D^{k+1}G_n^{(\gamma)}(1)}{D^k G_n^{(\gamma)}(1)} \geq 1 \quad k = 0 \ldots n-1 \tag{5.8}$$

and since both derivatives are positive, the lemma follows. □

We now have all the tools to prove

THEOREM 5.7. *The Gegenbauer tau approximation to problem (1.2) has real, negative, and distinct eigenvalues for* $1/2 < \gamma \leq 7/2$. *This* γ *range is sharp, spurious positive eigenvalues exist for* $\gamma < 1/2$ *and complex eigenvalues arise for* $7/2 < \gamma$.

Proof. This has already been proven in section 5.2 for even solutions. For odd solutions, we need to consider two separate cases.

Case 1. $3/2 < \gamma \leq 7/2$. The characteristic polynomial (4.11)

$$\sum_{k=0}^{\infty} \mu^k D^{2k} G_n^{(\gamma-2)}(1) - \sum_{k=0}^{\infty} \mu^k D^{2k+1} G_n^{(\gamma-2)}(1) = \Omega_n^{(\gamma-2)}(\mu) - \Theta_n^{(\gamma-2)}(\mu) \tag{5.9}$$

is a linear combination of the polynomials $\Omega_n^{(\gamma-2)}(\mu)$ and $\Theta_n^{(\gamma-2)}(\mu)$, which form a positive pair for $3/2 < \gamma \leq 7/2$, by theorem 5.4. Therefore, by theorem 5.3 this characteristic polynomial has real roots. Then by lemma 5.6 we deduce that all its coefficients are of the same sign, hence all its roots must be negative.

Case 2. $1/2 < \gamma \leq 3/2$. The polynomial

$$p_{n-1}^{(\gamma-1)}(z) = \Lambda(z^2) + z\Phi(z^2) \tag{5.10}$$

with $p_n^{(\gamma)}(z)$ as in theorem 5.5, is stable for the desired range of parameters by theorem 5.5, so the Hermite Biehler theorem (theorem 5.1) implies that the polynomials

$$\begin{aligned} \Lambda(\mu) &= \frac{G_{n-2}^{(\gamma-1)}(1) - G_n^{(\gamma-1)}(1)}{2(n+\gamma-2)} + \sum_{k=0}^{\infty} \mu^k\, D^{2k} G_{n-1}^{(\gamma-1)}(1) \\ &= \frac{G_{n-2}^{(\gamma-1)}(1) - G_n^{(\gamma-1)}(1)}{2(n+\gamma-2)} + \Omega_{n-1}^{(\gamma-1)}(\mu), \end{aligned} \tag{5.11}$$

and

$$\Phi(\mu) = \sum_{k=0}^{\infty} \mu^k\, D^{2k+1} G_{n-1}^{(\gamma-1)}(1) = \Theta_{n-1}^{(\gamma-1)}(\mu) \tag{5.12}$$

form a positive pair, with $\Omega_n^{(\gamma)}(\mu)$ and $\Theta_n^{(\gamma)}(\mu)$ as defined in theorem 5.4. Thus $\mu\Phi(\mu)$ and $\Lambda(\mu)$ have interlacing roots and any real linear combination of the two must have real roots (theorem 5.3). Now the characteristic polynomial (4.12) is in fact the linear combination $\mu\Phi(\mu) - \Lambda(\mu)$ so it has real roots. Its constant term is equal to

$$\frac{G_n^{(\gamma-1)}(1) - G_{n-2}^{(\gamma-1)}(1)}{2(n+\gamma-2)} - G_{n-1}^{(\gamma-1)}(1) \tag{5.13}$$

which is negative if $-1/2 < \gamma - 1 \le 1/2$, that is $1/2 < \gamma \le 3/2$, from (A.11). All other coefficients of the characteristic polynomial $\mu\Phi(\mu) - \Lambda(\mu)$ are negative by lemma 5.6. Since all coefficients have the same sign and all roots are real, all the roots must be negative. □

6. Galerkin and Modified tau methods. Our main theorem 5.7 can be expressed in terms of the inviscid Galerkin and Galerkin methods since these methods are equivalent to Gegenbauer tau methods with index $\gamma+1$ and $\gamma+2$, respectively, as shown in section 2.

COROLLARY 6.1. *The Gegenbauer Inviscid Galerkin approximation to problem (1.2) has real, negative, and distinct eigenvalues for* $-1/2 < \gamma \le 5/2$.

COROLLARY 6.2. *The Gegenbauer Galerkin approximation to problem (1.2) has real negative eigenvalues for* $-1/2 < \gamma \le 3/2$.

COROLLARY 6.3. *Since Chebyshev corresponds to* $\gamma = 0$ *and Legendre to* $\gamma = 1/2$*, the Chebyshev and Legendre tau approximations to problem (1.2) have spurious eigenvalues, but the Chebyshev and Legendre inviscid Galerkin (*$\gamma = 1$ *and* $3/2$*, respectively) and Galerkin (*$\gamma = 2$ *and* $5/2$*, respectively) approximations provide real, negative, and distinct eigenvalues.*

Finally, we prove that the modified tau method introduced by Gottlieb and Orszag [11] and developed by various authors [8], [16] is equivalent to the Galerkin method. McFadden, Murray and Boisvert [16] have already shown the equivalence between the modified Chebyshev tau and the Chebyshev Galerkin methods. Our simpler proof generalizes their results to the Gegenbauer class of approximations.

The idea for the modified tau method, widely used for time-marching, starts with the substitution $v(x) = D^2 u(x)$. Problem (1.2) reads

$$D^2 u = v, \qquad D^2 v = \lambda v, \quad \text{with } u = Du = 0 \ \text{ at } x = \pm 1. \tag{6.1}$$

If we approximate $u(x)$ by a polynomial of degree n, then $v = D^2u$ suggests that v should be a polynomial of degree $n-2$, however the modified tau method approximates both $u(x)$ and $v(x)$ by polynomials of degree n,

$$\begin{aligned} u_n(x) &= \sum_{k=0}^{n} \hat{u}_k G_k^{(\gamma)}(x), \qquad & D^2u_n(x) &= \sum_{k=0}^{n-2} \hat{u}_k^{(2)} G_k^{(\gamma)}(x), \\ v_n(x) &= \sum_{k=0}^{n} \hat{v}_k G_k^{(\gamma)}(x), \qquad & D^2v_n(x) &= \sum_{k=0}^{n-2} \hat{v}_k^{(2)} G_k^{(\gamma)}(x), \end{aligned} \tag{6.2}$$

where the superscripts indicate the Gegenbauer coefficients of the corresponding derivatives. These can be expressed in terms of the Gegenbauer coefficients of the original function using (A.10) twice, as in the Chebyshev tau method [3], [17]. Hence there are $2n+2$ coefficients to be determined, $\hat{u}_0, \dots, \hat{u}_n$, $\hat{v}_0, \dots, \hat{v}_n$. In the modified tau method, these are determined by the 4 boundary conditions and the $2n-2$ tau equations obtained from orthogonalizing the residuals of both equations $D^2u = v$ and $D^2v = \lambda v$ to the first $n-1$ Gegenbauer polynomials $G_0^{(\gamma)}(x), \dots, G_{n-2}^{(\gamma)}(x)$ with respect to the Gegenbauer weight $(1-x^2)^{\gamma-1/2}$. In terms, of the expansions (6.2), these weighted residual equations have the simple form

$$\begin{aligned} \hat{u}_k^{(2)} &= \hat{v}_k, \qquad & 0 \le k \le n-2, \\ \hat{v}_k^{(2)} &= \lambda \hat{v}_k, \qquad & 0 \le k \le n-2. \end{aligned} \tag{6.3}$$

McFadden *et al.*[16] showed that the modified Chebyshev tau is equivalent to the Chebyshev Galerkin method for this particular problem. We provide a simpler proof for the general setting of Gegenbauer polynomials.

THEOREM 6.4. *The modified Gegenbauer tau method proposed in [11] is equivalent to the Gegenbauer Galerkin method for problem (1.2).*

Proof. Let the polynomial approximations and their derivatives as in (6.2). Then the tau equations (6.3) are equivalent to the residual equations

$$\begin{aligned} v_n(x) - D^2u_n(x) &= \hat{v}_{n-1} G_{n-1}^{(\gamma)}(x) &+ \hat{v}_n G_n^{(\gamma)}(x), \\ (\lambda - D^2)v_n(x) &= \lambda \hat{v}_{n-1} G_{n-1}^{(\gamma)}(x) &+ \lambda \hat{v}_n G_n^{(\gamma)}(x). \end{aligned} \tag{6.4}$$

Combining the two yields

$$(\lambda - D^2)D^2u_n(x) = \hat{v}_{n-1}\, D^2G_{n-1}^{(\gamma)}(x) + \hat{v}_n\, D^2G_n^{(\gamma)}(x), \quad u_n(\pm 1) = Du_n(\pm 1) = 0 \tag{6.5}$$

which using (A.5) is equivalent to

$$(\lambda - D^2)D^2u_n(x) = \tau_0 \lambda G_{n-2}^{(\gamma+2)}(x) + \tau_1 \lambda G_{n-3}^{(\gamma+2)}(x), \quad u_n(\pm 1) = Du_n(\pm 1) = 0. \tag{6.6}$$

with $\hat{v}_n = 4\lambda(\gamma+1)(\gamma+2)\tau_0$ and $\hat{v}_{n-1} = 4\lambda(\gamma+1)(\gamma+2)\tau_1$. This is exactly the Gegenbauer Galerkin method, as given in equation (2.5). ◻

Since the modified Gegenbauer tau is equivalent to the Galerkin method, the results in section 5 imply that the modified tau method for problem 1.2 has real and negative eigenvalues for $-1/2 < \gamma \le 3/2$. This includes Chebyshev for $\gamma = 0$ and Legendre for $\gamma = 1/2$.

Acknowledgments. We thank Jue Wang for helpful calculations in the early stages of this work. This research was partially supported by NSF grant DMS-0204636.

Appendix A. Gegenbauer (Ultraspherical) Polynomials.

The Gegenbauer (a.k.a. Ultraspherical) polynomials $C_n^{(\gamma)}(x)$, $\gamma > -1/2$, of degree n are the Jacobi polynomials with $\alpha = \beta = \gamma - 1/2$, up to normalization [1, 22.5.20]. They are symmetric (even for n even and odd for n odd) orthogonal polynomials with weight function $W^{(\gamma)}(x) = (1-x^2)^{\gamma-1/2}$. Since the standard normalization [1, 22.3.4], is singular for the Chebyshev case $\gamma = 0$, we use a non-standard normalization that includes the Chebyshev case but preserves the simplicity of the Gegenbauer recurrences. Set

$$G_0^{(\gamma)}(x) := 1, \qquad G_n^{(\gamma)}(x) := \frac{C_n^{(\gamma)}(x)}{2\gamma}, \quad n \geq 1. \tag{A.1}$$

These Gegenbauer polynomials satisfy the orthogonality relationship

$$\int_{-1}^{1} (1-x^2)^{\gamma-1/2}\, G_m^{(\gamma)} G_n^{(\gamma)}\, dx = \begin{cases} 0, & m \neq n, \\ h_n^\gamma, & m = n, \end{cases} \tag{A.2}$$

where [1, 22.2.3],

$$h_0^\gamma = \frac{\pi 2^{-2\gamma}\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)} \qquad \text{and} \qquad h_n^\gamma = \frac{\pi 2^{-1-2\gamma}\Gamma(n+2\gamma)}{(n+\gamma)n!\Gamma^2(\gamma+1)} \qquad n \geq 1. \tag{A.3}$$

The Sturm-Liouville form of the Gegenbauer equation for $G_n^{(\gamma)}(x)$ is

$$D\left[(1-x^2)^{\gamma+1/2}\, DG_n^{(\gamma)}(x)\right] = -n(n+2\gamma)\,(1-x^2)^{\gamma-1/2}\, G_n^{(\gamma)}(x) \tag{A.4}$$

They satisfy the derivative recurrence formula

$$\frac{d}{dx} G_{n+1}^{(\gamma)} = 2(\gamma+1) G_n^{(\gamma+1)}, \tag{A.5}$$

(for $C_n^{(\gamma)}$ this is formula [2, A.57]), and their three-term recurrence takes the simple form

$$(n+1)G_{n+1}^{(\gamma)} = 2(n+\gamma)xG_n^{(\gamma)} - (n-1+2\gamma)G_{n-1}^{(\gamma)}, \quad n \geq 2,, \tag{A.6}$$

with

$$G_0^{(\gamma)}(x) = 1, \quad G_1^{(\gamma)}(x) = x, \quad G_2^{(\gamma)} = (\gamma+1)x^2 - \frac{1}{2}. \tag{A.7}$$

These recurrences can be used to verify the odd-even symmetry of Gegenbauer polynomials [1, 22.4.2]

$$G_n^{(\gamma)}(x) = (-1)^n G_n^{(\gamma)}(-x). \tag{A.8}$$

Differentiating the recurrence (A.6) with respect to x and subtracting from the corresponding recurrence for $\gamma + 1$ using (A.5), yields [1, 22.7.23]

$$(n+\gamma)G_n^{(\gamma)} = (\gamma+1)\left[G_n^{(\gamma+1)} - G_{n-2}^{(\gamma+1)}\right], \qquad n \geq 3. \tag{A.9}$$

Combined with (A.5), this leads to the important derivative recurrence between Gegenbauer polynomials of same index γ

$$
\begin{aligned}
&G_0^{(\gamma)}(x) = DG_1^{(\gamma)}(x), \quad 2(1+\gamma)G_1^{(\gamma)}(x) = DG_2^{(\gamma)}(x),\\
&2(n+\gamma)G_n^{(\gamma)} = \frac{d}{dx}\left[G_{n+1}^{(\gamma)} - G_{n-1}^{(\gamma)}\right].
\end{aligned} \tag{A.10}
$$

Evaluating the Gegenbauer polynomial at $x = 1$ we find [1, 22.4.2],
(A.11)

$$
\begin{aligned}
G_n^{(\gamma)}(1) = \frac{1}{2\gamma}C_n^{(\gamma)}(1) = \frac{1}{2\gamma}\binom{2\gamma+n-1}{n} &= \frac{(2\gamma+n-1)(2\gamma+n-2)\cdots(2\gamma+1)}{n!}\\
&= \frac{\Gamma(2\gamma+n)}{n!\,\Gamma(2\gamma+1)}
\end{aligned}
$$

for $n \geq 2$, with $G_1^{(\gamma)}(1) = G_0^{(\gamma)}(1) = 1$, where $\Gamma(z)$ is the standard gamma function [1]. Note that $G_n^{(\gamma)}(1) > 0$ for $\gamma > -1/2$ and that it decreases with increasing n if $-1/2 < \gamma < 1/2$ but increases with n if $1/2 < \gamma$.

Now (A.5) gives

$$
\frac{d^k G_n^{(\gamma)}}{dx^k}(x) = \frac{2^k\Gamma(\gamma+k+1)}{\Gamma(\gamma+1)}G_{n-k}^{(\gamma+k)}(x), \tag{A.12}
$$

which coupled with (A.11) gives

$$
\frac{d^k G_n^{(\gamma)}}{dx^k}(1) = \frac{2^{k-1}\Gamma(\gamma+k)\Gamma(n+2\gamma+k)}{(n-k)!\Gamma(\gamma+1)\Gamma(2\gamma+2k)}, \qquad n \geq 1. \tag{A.13}
$$

Gegenbauer polynomials correspond to Chebyshev polynomials of the 1st kind, $T_n(x)$, when $\gamma = 0$, to Legendre $P_n(x)$ for $\gamma = 1/2$ and to Chebyshev of the 2nd kind, $U_n(x)$, for $\gamma = 1$. For the non standard normalization,

$$
G_n^{(0)}(x) = \frac{T_n(x)}{n}, \quad G_n^{(1/2)}(x) = P_n(x), \quad G_n^{(1)}(x) = \frac{U_n(x)}{2}. \tag{A.14}
$$

Appendix B. Integrals and Asymptotics. As shown in section 3, the tau equations (3.1) provide a matrix eigenproblem of the form $\mu Aa = Ba$. To reduce the coefficients $A(k,l)$ and $B(k,l)$ defined in (3.2) and (3.3) to the expressions (3.4) and (3.5), use (A.4) and (A.5) repeatedly

$$
\begin{aligned}
\text{(B.1)}\quad D^2\left[(1-x^2)^2G_l^{(5/2)}(x)\right] &= \frac{1}{5}D^2\left[(1-x^2)^2DG_{l+1}^{(3/2)}(x)\right]\\
= -\frac{1}{5}(l+1)(l+4)D\left[(1-x^2)G_{l+1}^{(3/2)}(x)\right] &= -\frac{1}{15}(l+1)(l+4)D\left[(1-x^2)DG_{l+2}^{(1/2)}(x)\right]\\
&= \frac{1}{15}(l+1)(l+2)(l+3)(l+4)\,G_{l+2}^{(1/2)}(x) \equiv \mathcal{C}_l\,G_{l+2}^{(1/2)}(x).
\end{aligned}
$$

For the perturbation analysis described in section 3 we need the first order corrections to $B(0,n-4)$ and to $B(1,n-5)$. From equations (3.12) and (3.13)

$$
B_1(0,n-4) = \lim_{\epsilon\to 0}\frac{\mathcal{C}_{n-4}}{\epsilon}\int_{-1}^{1} P_{n-2}(x)\,(1-x^2)^\epsilon\,dx, \tag{B.2}
$$

$$
B_1(1,n-5) = \lim_{\epsilon\to 0}\frac{\mathcal{C}_{n-5}}{\epsilon}\int_{-1}^{1} x\,P_{n-3}(x)\,(1-x^2)^\epsilon\,dx. \tag{B.3}
$$

To evaluate $\int_{-1}^{1} P_n(x)(1-x^2)^\epsilon dx$ to $O(\epsilon)$ for $|\epsilon| \ll 1$ and n even, use (A.4) for $\gamma = 1/2$ and integration by parts to derive

$$\begin{aligned}\int_{-1}^{1} P_n(x)(1-x^2)^\epsilon dx =& \frac{-1}{n(n+1)} \int_{-1}^{1} D\left[(1-x^2)DP_n\right](1-x^2)^\epsilon dx \\ =& \frac{\epsilon}{n(n+1)} \int_{-1}^{1} DP_n\,(1-x^2)^\epsilon(-2x)dx.\end{aligned} \tag{B.4}$$

This integral is 0 if n is odd since $P_n(x) = (-1)^n P_n(-x)$. Since we have an ϵ prefactor we can now set $\epsilon = 0$ in the integral, and do the remaining integral by parts to obtain

$$\begin{aligned}\int_{-1}^{1} P_n(x)(1-x^2)^\epsilon dx \sim& \frac{-2\epsilon}{n(n+1)} \int_{-1}^{1} xDP_n dx \\ =& \frac{-2\epsilon}{n(n+1)} \int_{-1}^{1} \left(D(xP_n) - P_n\right) dx \\ =& \frac{-2\epsilon}{n(n+1)} \left(P_n(1) + P_n(-1)\right) = \frac{-4\epsilon}{n(n+1)}\end{aligned} \tag{B.5}$$

for n even (0 for n odd as should be).

For the integral in (B.3), use the recurrence (A.6) for $\gamma = 1/2$ to write $(2n-5)xP_{n-3}(x) = (n-2)P_{n-2}(x) + (n-3)P_{n-4}(x)$ and evaluate the resulting 2 integrals from the formula (B.5). Hence, for n even,

$$B_1(0, n-4) \sim -\frac{4\mathcal{C}_{n-4}}{(n-2)(n-1)}, \tag{B.6}$$

$$B_1(1, n-5) \sim -\frac{4\mathcal{C}_{n-5}}{(n-4)(n-1)}. \tag{B.7}$$

Now for $A_0(0, n-4)$ and $A_0(1, n-5)$ and n even, we have

$$A_0(0, n-4) = \mathcal{C}_{n-4} \int_{-1}^{1} D^2 P_{n-2}(x)dx = (n-2)(n-1)\mathcal{C}_{n-4}, \tag{B.8}$$

$$A_0(1, n-5) = \mathcal{C}_{n-5} \int_{-1}^{1} xD^2 P_{n-3}(x)dx = (n-4)(n-1)\mathcal{C}_{n-5}. \tag{B.9}$$

Appendix C. Proof of theorem 5.5.

Consider

$$f_n(x; z) = \sum_{k=0}^{\infty} z^k D^k G_n^{(\gamma)}(x) + K\left(G_n^{(\gamma)}(x) - G_{n+2}^{(\gamma)}(x)\right) \tag{C.1}$$

where z is a solution of $f_n(1; z) = 0$ and K is

$$K = \frac{(G_{n-1}^{(\gamma)}(1) - G_{n+1}^{(\gamma)}(1))}{2(n+\gamma)(G_n^{(\gamma)}(1) - G_{n+2}^{(\gamma)}(1))} = \ldots = \frac{n+2}{2(n+\gamma+1)(n+2\gamma-1)} \tag{C.2}$$

where we have used (A.11). Note that $f_n(1; z) = p_n^{(\gamma)}(z)$ defined in theorem 5.5. Taking the x-derivative of $f_n(x; z)$ and using (A.10), we find

$$\frac{df_n(x; z)}{dx} = \sum_{k=0}^{\infty} z^k D^{k+1} G_n^{(\gamma)}(x) - 2K(n+\gamma+1)G_{n+1}^{(\gamma)}(x). \tag{C.3}$$

Thus $f_n(x;z)$ satisfies the following differential equation

$$f_n(x;z)-(1+K)G_n^{(\gamma)}(x)+KG_{n+2}^{(\gamma)}(x)=z\frac{df_n(x;z)}{dx}+z2K(n+\gamma+1)G_{n+1}^{(\gamma)}(x). \tag{C.4}$$

Multiplying by $(1+x)\frac{df_n^*(x;z)}{dx}$, integrating in the Gegenbauer norm and adding the complex conjugate we get

$$\begin{aligned}\int_{-1}^{1}\frac{d|f_n|^2}{dx}(1+x)w(x)dx-(1+K)\left(\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_n^{(\gamma)}(x)w(x)dx+C.C.\right)\\+K\left(\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+2}^{(\gamma)}(x)w(x)dx+C.C.\right)=(z+z^*)\int_{-1}^{1}\left|\frac{df_n}{dx}\right|^2(1+x)w(x)dx\\+\left(z2K(n+\gamma+1)\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+1}^{(\gamma)}(x)w(x)dx+C.C.\right)\end{aligned} \tag{C.5}$$

where $C.C.$ denotes the complex conjugate. To simplify (C.5) we need to compute four simple integrals

$$\begin{aligned}I_1&=\int_{-1}^{1}\frac{d|f_n|^2}{dx}(1+x)w(x)dx\\J_0&=\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_n^{(\gamma)}(x)w(x)dx\\J_1&=\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+1}^{(\gamma)}(x)w(x)dx\\J_2&=\int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+2}^{(\gamma)}(x)w(x)dx.\end{aligned} \tag{C.6}$$

Using integration by parts, the first integral becomes

$$I_1=\int_{-1}^{1}\frac{d|f_n|^2}{dx}(1+x)w(x)dx=-\int_{-1}^{1}|f_n|^2(1-2\gamma x)\frac{w(x)}{1-x}dx. \tag{C.7}$$

Therefore, the integral is negative for $-1/2<\gamma\le 1/2$ since for this range of parameters $1-2\gamma x$ is positive.

For the calculation of the three other integrals we are going to need the expression

$$\frac{df_n}{dx}=\sum_{k=0}^{\infty}z^kD^{k+1}G_n^{(\gamma)}(x)-2K(\gamma+n+1)G_{n+1}^{(\gamma)}(x) \tag{C.8}$$

$$=\mathcal{P}_{n-2}(x;z)+2(\gamma+n-1)G_{n-1}^{(\gamma)}(x)-2K(\gamma+n+1)G_{n+1}^{(\gamma)}(x) \tag{C.9}$$

$$=\mathcal{P}_{n-1}(x;z)-2K(\gamma+n+1)G_{n+1}^{(\gamma)}(x) \tag{C.10}$$

where $\mathcal{P}_{n-2}(x;z)$ and $\mathcal{P}_{n-1}(x;z)$ are polynomials of degree $n-2$ and $n-1$, respectively.

With the use of (C.9) and orthogonality of the Gegenbauer polynomials, we find

$$\text{(C.11)}\quad J_0 = \int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_n^{(\gamma)}(x)w(x)dx =$$
$$2(n-1+\gamma)\int_{-1}^{1}xG_{n-1}^{(\gamma)}(x)G_n^{(\gamma)}(x)w(x)dx - 2K(n+1+\gamma)\int_{-1}^{1}xG_{n+1}^{(\gamma)}(x)G_n^{(\gamma)}(x)w(x)dx =$$
$$n\int_{-1}^{1}(G_n^{(\gamma)}(x))^2w(x)dx - K(n+\gamma+1)\frac{n+1}{n+\gamma}\int_{-1}^{1}(G_{n+1}^{(\gamma)}(x))^2w(x)dx =$$
$$nh_n^{(\gamma)} - K\frac{(n+\gamma+1)(n+1)}{n+\gamma}h_{n+1}^{(\gamma)}$$

where we have also used (A.5) and (A.6). In the same way we compute

$$\text{(C.12)}\quad J_2 = \int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+2}^{(\gamma)}(x)w(x)dx =$$
$$-K\int_{-1}^{1}2(n+\gamma+1)xG_{n+1}^{(\gamma)}(x)G_{n+2}^{(\gamma)}(x)w(x)dx = -K(n+2)h_{n+2}^{(\gamma)}$$

and

(C.13)
$$J_1 = \int_{-1}^{1}(1+x)\frac{df_n^*}{dx}G_{n+1}^{(\gamma)}(x)w(x)dx = -2K(n+\gamma+1)\int_{-1}^{1}(1+x)(G_{n+1}^{(\gamma)}(x))^2w(x)dx =$$
$$-2K(n+\gamma+1)\int_{-1}^{1}(G_{n+1}^{(\gamma)}(x))^2w(x)dx = -2K(n+\gamma+1)h_{n+1}^{(\gamma)}.$$

Thus, (C.5) transforms to

(C.14)
$$-\int_{-1}^{1}|f_n|^2\frac{(1-2\gamma x)w(x)}{(1-x)}dx - 2(1+K)\left(nh_n^{(\gamma)} - K\frac{(n+\gamma+1)(n+1)}{n+\gamma}h_{n+1}^{(\gamma)}\right)$$
$$-2K^2(n+2)h_{n+2}^{(\gamma)} = (z+z^*)\left(\int_{-1}^{1}\left|\frac{df_n}{dx}\right|^2(1+x)w(x)dx - 4K^2(n+1+\gamma)^2h_{n+1}^{(\gamma)}\right).$$

Our task is to show that the left hand side of the above expression is negative whereas the coefficient of the term $z+z^*$ on the right hand side is positive. A simple calculation shows that

$$\text{(C.15)}\quad nh_n^{(\gamma)} - K\frac{(n+\gamma+1)(n+1)}{(n+\gamma)}h_{n+1}^{(\gamma)} = nh_n^{(\gamma)} - \frac{(n+1)(n+2)}{(n+\gamma)(n+2\gamma-1)}h_{n+1}^{(\gamma)}$$
$$= \frac{\pi 2^{-1-2\gamma}\Gamma(n+2\gamma)}{\gamma^2\Gamma^2(\gamma)n!(n+\gamma)}\left(n - \frac{(n+2)(n+2\gamma)}{2(n-1+2\gamma)(n+1+\gamma)}\right)$$
$$\geq \frac{\pi 2^{-1-2\gamma}\Gamma(n+2\gamma)}{\gamma^2\Gamma^2(\gamma)n!(n+\gamma)}\left(n - \frac{(n+2)(n+1)}{(n-2)(2n+1)}\right)$$

The last parenthesis is positive for $n \geq 3$.

For the right hand side we use the notation in (C.10) to get

$$\int_{-1}^{1} \left|\frac{df_n}{dx}\right|^2 (1+x)w(x)dx - 4K^2(n+1+\gamma)^2 h_{n+1}^{(\gamma)} = \tag{C.16}$$

$$\int_{-1}^{1}(1+x)\Big(|\mathcal{P}_{n-1}(x;z)|^2 - 2K(n+1+\gamma)G_{n+1}^{(\gamma)}(x)\left(\mathcal{P}_{n-1}(x;z)+\mathcal{P}_{n-1}^*(x;z)\right)$$

$$+4K^2(n+1+\gamma)^2(G_{n+1}^{(\gamma)}(x))^2\Big)w(x)dx - 4K^2(n+1+\gamma)^2 h_{n+1}^{(\gamma)} =$$

$$\int_{-1}^{1}(1+x)|\mathcal{P}_{n-1}(x;z)|^2 w(x)dx + 0 + 0 + 4K^2(n+1+\gamma)^2 h_{n+1}^{(\gamma)} - 4K^2(n+1+\gamma)^2 h_{n+1}^{(\gamma)}$$

$$= \int_{-1}^{1}(1+x)\left|\mathcal{P}_{n-1}(x;z)\right|^2 w(x)dx > 0.$$

Thus (C.14) becomes

(C.17)

$$-\int_{-1}^{1}|f_n|^2\frac{(1-2\gamma x)w(x)}{(1-x)}dx - 2(1+K)\left(nh_n^{(\gamma)} - K\frac{(n+\gamma+1)(n+1)}{n+\gamma}h_{n+1}^{(\gamma)}\right)$$

$$-2K^2(n+2)h_{n+2}^{(\gamma)} = (z+z^*)\int_{-1}^{1}(1+x)\left|\mathcal{P}_{n-1}(x;z)\right|^2 w(x)dx$$

and $\Re(z) < 0$.

For $n = 2$ we get the following characteristic polynomial

$$f_2^{(\gamma)}(1;z) = \frac{2}{3}(\gamma+1)(3z^2+3z+1) \tag{C.18}$$

whose zeros

$$z_1 = -\frac{1}{2} - \frac{\sqrt{3}i}{6}, \qquad z_2 = -\frac{1}{2} + \frac{\sqrt{3}i}{6} \tag{C.19}$$

have negative real parts for any γ.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] John P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover, New York, 2001.

[3] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral Methods in Fluid Dynamics*, Springer, New York, 1988.

[4] M. Charalambides and F. Waleffe, *Spectrum of the Jacobi tau approximation for the second derivative operator*, SIAM J. Numer. Anal., 46 (2008), pp. 280–294.

[5] G. Csordas, M. Charalambides, and F. Waleffe, *A new property of a class of Jacobi polynomials*, Proc. AMS, 133 (2005), pp. 3551–3560.

[6] Paul T. Dawkins, Steven R. Dunbar, and Rod W. Douglass, *The origin and nature of spurious eigenvalues in the spectral tau method*, J. Comput. Phys., 147 (1998), pp. 441–462.

[7] P. Drazin and W.H. Reid, *Hydrodynamic Stability*, Cambridge University Press, Cambridge, UK, 1981.

[8] David R. Gadner, Steven A. Trogdon, and Rod W. Douglass, *A modified tau spectral method that eliminates spurious eigenvalues*, J. Comput. Phys., 80 (1989), pp. 137–167.

[9] D. Gottlieb, *The stability of pseudospectral-Chebyshev methods*, Math. Comp., 36 (1981), pp. 107–118.

[10] D. Gottlieb and L. Lustman, *The spectrum of the Chebyshev collocation operator for the heat equation*, SIAM J. Numer. Anal., 20 (1983), pp. 909–921.

[11] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, SIAM, Philadelphia, 1977.

[12] O. Holtz, *Hermite-Biehler, Routh-Hurwitz, and total positivity*, Linear Algebra Appl., 372 (2003), pp. 105–110.

[13] Weizhang Huang and David M. Sloan, *The pseudospectral method for solving differential eigenvalue problems*, J. Comput. Phys., 111 (1994), pp. 399–409.

[14] J. Kim, P. Moin, and R. Moser, *Turbulence statistics in fully developed channel flow at low reynolds number*, J. Fluid Mech., 177 (1987), pp. 133–166.

[15] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, AMS, Providence, 1980.

[16] G. B. McFadden, B. T. Murray, and R. F. Boisvert, *Elimination of spurious eigenvalues in the Chebyshev tau spectral method*, J. Comput. Phys., 91 (1990), pp. 228–239.

[17] S. A. Orszag, *Accurate solution of the Orr-Sommerfeld stability equation*, Journal of Fluid Mechanics, 50 (1971), pp. 689–703.

[18] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, Oxford Univ. Press Inc, New York, 2002.

[19] F. Waleffe, *Exact coherent structures in channel flow*, J. Fluid Mech., 435 (2001), pp. 93–102.

[20] F. Waleffe, *Homotopy of exact coherent structures in plane shear flows*, Phys. Fluids, 15 (2003), pp. 1517–1534.

[21] J. Wang, J. Gibson, and F. Waleffe, *Lower branch coherent states in shear flows: transition and control*, Phys. Rev. Lett., 98 (2007), p. 204501.

[22] A. Zebib, *Removal of spurious modes encountered in solving stability problems by spectral methods*, J. Comp. Phys., 70 (1987), pp. 521–525.